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2001 J. Phys. A: Math. Gen. 34 9641

(http://iopscience.iop.org/0305-4470/34/45/304)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 9641–9651 PII: S0305-4470(01)22862-5

The q deformation of AKNS-D hierarchy

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Received 5 March 2001, in final form 14 August 2001
Published 2 November 2001
Online at stacks.iop.org/JPhysA/34/9641

Abstract

In this paper, we have considered the q deformation of the AKNS-D hierarchy, proved the bilinear identity and obtained the τ function of the q deformed AKNS-D hierarchy.

PACS numbers: 02.30.-f, 02.30.Ik, 05.45.Yv

1. Introduction

Recently, more and more attention has been paid to the discrete integrable system. For example, one can find good reviews and a large number of references about discrete integrable systems in the book edited by Bobenko and Seiler [1]. One topic discussed is the semi-discretization of an integrable system. It is an integrable system with many variables and one discrete variable. There are lot of discussions about the q deformation of KdV and the KdV hierarchy, q deformation of the KP hierarchy, their solutions and τ functions as well as other properties [2–12], where τ functions of the q KP hierarchy [9, 10] could be constructed from the τ functions of the classical KP hierarchy by making an appropriate shift. In this paper, we consider the AKNS-D hierarchy which was proposed by Dickey [14] and its q deformation.

This paper is organized as follows: in section 2, we will give a brief review of the AKNS-D hierarchy to make this paper self-contained. In section 3, we will give the definition of the q AKNS-D hierarchy, their Baker functions and the bilinear identity. The derivation of the τ functions is discussed in section 4.

2. Brief review of the AKNS-D hierarchy

In this section, we briefly review the general idea of the AKNS-D hierarchy. The details can be found in [13–15].

Let

$$L = \partial + U - zA \tag{1}$$

where $\partial = \frac{\partial}{\partial x}$, $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$, U is a matrix function of x with $u_{ii} = 0$ and rank n. The resolvent of L is defined as a series:

$$R = \sum_{i=0}^{\infty} R^{(i)} z^{-i} \tag{2}$$

which commutes with L, i.e. $[L, R] = 0$.

The elements of $R^{(i)}$ are all differential polynomials of u_{ij}. The set of all resolvents forms an n-dim algebra over the field of constant diagonal series $C(z) = \sum_{i=0}^{\infty} C_i z^{-i}$.The basis of resolvents are R_α

$$R_\alpha = E_\alpha + \sum_{j=1}^{\infty} R_\alpha^{(j)} z^{-j} \tag{3}$$

where E_α is the matrix with the only non-zero element at the $\alpha\alpha$ position. All elements of $R_\alpha^{(j)}$, $j > 0$, are differential polynomials without constant terms. The basic resolvents satisfy the relation $R_\alpha R_\beta = \delta_{\alpha\beta} R_\beta$. Take

$$B_{k\alpha} = (z^k R_\alpha)_+ = \sum_{j=0}^{k} R_\alpha^{(j)} z^{k-j}. \tag{4}$$

The subscript + means taking non-negative powers of z. The AKNS hierarchy is the set of equations

$$\partial_{k\alpha} L = [B_{k\alpha}, L] \tag{5}$$

where $\partial_{k\alpha}$ means $\frac{\partial}{\partial t_{k\alpha}}$, and $t_{k\alpha}, k = 0, 1, 2, \ldots, \alpha = 1, 2, \ldots, n$ is a set of time variables. In these equations, the operators ∂ and $\partial_{k\alpha}$ are not independent. They have the following relation [13]:

$$\partial = \sum_{\alpha=1}^{n} a_\alpha \partial_{1\alpha}. \tag{6}$$

Denote the wavefunction by

$$\hat{w}(z) = I + \sum_{j=1}^{\infty} w_j z^{-j}. \tag{7}$$

It is well known that the operator L can be represented in the following form:

$$L = \hat{w}(z)(\partial - zA)\hat{w}(z)^{-1}. \tag{8}$$

Then the basic resolvents are given by

$$R_\alpha = \hat{w}(z) E_\alpha \hat{w}(z)^{-1} \tag{9}$$

and the formal Baker function is

$$w = \hat{w}(z) \exp\left(\sum_{k=0}^{\infty} \sum_{\alpha=0}^{n} z^k E_\alpha t_{k\alpha} \right). \tag{10}$$

With the formal Baker function, the operator L and the resolvent can be written as

$$L = w \partial w^{-1} \tag{11}$$

$$R_\alpha = w E_\alpha w^{-1}. \tag{12}$$

The equations of the hierarchy are equivalent to

$$\begin{aligned} &L(w) = 0 \\ &\partial_{k\alpha} w = B_{k\alpha} w \qquad \text{or} \qquad \partial_{k\alpha} \hat{w} = -(z^k R_\alpha)_- \hat{w}. \end{aligned} \tag{13}$$

To avoid some possible confusion, from now on we use $L(f)$ or (Lf) to denote an operator L acting on a function f and use Lf to note multiplication by two operator.

The adjoint Baker function is defined as $w^* = (w^{-1})^T$ which satisfies the adjoint equation

$$L^*(w^*) = 0 \tag{14}$$

where $L^* = -\partial + (U - zA)^T$.

Proposition 1 (Bilinear identity). *A bilinear relation*

$$\mathrm{res}_z[z^l(\partial_{k_1\alpha_1} \ldots \partial_{k_s\alpha_s} w)(w^*)^T] = 0$$

holds, where $l = 0, 1, \ldots,$ *and* $(k_1\alpha_1), \ldots, (k_s, \alpha_s)$ *is any set of indices. Conversely, let two functions*

$$w = \left(I + \sum_{i=1}^{\infty} w_i z^{-\mathrm{i}}\right) \exp\left(\sum_{k=0}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right)$$

$$w^* = \left(I + \sum_{i=1}^{\infty} w_i^* z^{-\mathrm{i}}\right) \exp\left(-\sum_{k=0}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right)$$

satisfy the above bilinear identity. Then w *and* w^* *are, respectively, the Baker function and the adjoint Baker function of an operator* L *which satisfies the hierarchy equation.*

3. q AKNS-D hierarchy

3.1. Some useful results of q *calculation*

In this section, we will give some basic definitions for the q difference calculation and some useful relations for the calculation, without proof. Details can be found in [10, 17].

First, we introduce two operators:

$$Df(x) \ := f(qx) \tag{15}$$

$$D_q f(x) := \frac{f(qx) - f(x)}{x(q-1)}. \tag{16}$$

The first difference between the q difference and ordinary differential calculus is Leibnitz's law. The q Leibnitz law is

$$\begin{aligned} D_q(fg) &= (Df)\cdot(D_q g) + (D_q f)\cdot g \\ &= f\cdot(D_q g) + (D_q f)\cdot(Dg). \end{aligned} \tag{17}$$

Using this Leibnitz law, it is easy to show the following lemma:

Lemma 1.

$$D_q^n(fg) = \sum_{k=0}^{\infty} (C_k^n)_q (D^{n-k} D_q^k f) D_q^{n-k} g \tag{18}$$

$$D_q^m D_q^n f = D_q^{m+n} f \tag{19}$$

where $(C_k^n)_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}$ *and* $(C_0^n)_q = 1$.

Another useful fact about the q difference is the q exponential function. It is defined as

$$\exp_q(x) := \sum_{k=0}^{\infty} \frac{(1-q)^k}{(q;q)_k} x^k \tag{20}$$

where $(a;q)_k := \prod_{s=0}^{k-1}(1-aq^s)$ and $(a;q)_0 = 1$. A useful feature of this function is that the behaviour of $\exp_q(x)$ acted on by the q difference operator is just like the exponential function acted on by the ordinary differential operator, i.e.

$$D_q \exp_q(zx) = z \exp_q(zx). \tag{21}$$

Two other useful relations for $\exp_q(x)$ are

$$\exp_q(x) = \exp\left(\sum_{k=1}^{\infty} \frac{(1-q)^k}{k(1-q^k)} x^k\right) \tag{22}$$

$$(\exp_q(x))^{-1} = \exp_{1/q}(-x). \tag{23}$$

Using the above relation, direct calculation gives

Lemma 2.

$$\exp_q(zAx) \exp\left(\sum_{k=0}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right) = \exp\left(\sum_{k=0}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha {t'}_{k\alpha}\right)$$

where ${t'}_{k\alpha} = t_{k\alpha} + \frac{(1-q)^k}{k(1-q^k)}(a_\alpha x)^k$ *and* $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$.

For later convenience, we define the q commutator $[,]_q$ as

$$[A, B]_q = (DA)\cdot B - B \cdot A. \tag{24}$$

This bracket can be seen as the q deformation of the ordinary commutator. The operator D comes from the q Leibnitz law.

Introduce a L^2-metric on the space of n-dim matrix functions as

$$\langle A, B\rangle := \mathrm{tr}\int_{-\infty}^{+\infty} A\cdot B\,\mathrm{d}x. \tag{25}$$

Using this inner product, we can define the dual operator as usual:

$$\langle f, (g)P^*\rangle := \langle P(f), g\rangle. \tag{26}$$

It is easy to show that $(D_q)^* = (-\frac{1}{q})D_{1/q}$.

3.2. q AKNS-D hierarchy

Let $L_q = D_q - zA + U$, where $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$, U is an n-dim matrix function of x with $u_{ii} = 0$, for any i. Like the AKNS-D hierarchy, we define the resolvent R for L_q as

$$R = \sum_{i=0}^{\infty} R(i) z^{-i} \qquad [R, L_q]_q = 0 \tag{27}$$

i.e.

$$D_q R - [R, (U - zA)]_q = 0. \tag{28}$$

Submitting the formal expression of R into equation (28), we can get

$$D_q R^{(j)} - [R^{(j)}, U]_q + [R^{(j+1)}, A]_q = 0 \tag{29}$$

$$[R^{(0)}, A]_q = 0. \tag{30}$$

Lemma 3. *The set of all resolvents R form an algebra over the field of the formal series* $c(z) = \sum_{i=0}^{\infty} c_i z^{-i}$ *and we denote it by* $\Re$.

Proof. First, it is easy to see that, if R^1, R^2 satisfy equation (28), then we have $[c_1(z)R^1 + c_2(z)R^2, L_q]_q = 0$.

Second, if R^1, R^2 are two resolvents, then

$$[R^1R^2, L_q]_q = (DR^1)[R^2, L_q]_q + [R^1, L_q]_q R^2 = 0.$$

So the set of all resolvents form an algebra. □

We define the q wavefunction $\hat{w}$ as

$$\hat{w}_q := I + \sum_{k=1}^{\infty} w_k z^{-k} \tag{31}$$

which satisfies

$$L_q = (D\hat{w}_q) \cdot (D_q - zA) \cdot \hat{w}_q^{-1}. \tag{32}$$

The existence of $\hat{w}_q$ is obvious, because we can rewrite the above equation as

$$L_q \hat{w}_q = (D\hat{w}_q) \cdot (D_q - zA). \tag{33}$$

Using expression (31) and following the method in the classical case, i.e. the AKNS-D hierarchy, we can obtain these $\hat{w}_j$.

Lemma 4. *$R_\alpha = \hat{w}_q E_\alpha \hat{w}_q^{-1}$ is a resolvent which has the following property:*

$$R_\alpha \cdot R_\beta = \delta_{\alpha\beta} R_\beta.$$

Proof.

$$\begin{aligned}
[R_\alpha, L_q]_q &= (D\hat{w}_q)E_\alpha(D\hat{w}_q^{-1}) \cdot L_q - L_q \cdot \hat{w}_q E_\alpha \hat{w}_q^{-1} \\
&= (D\hat{w}_q)E_\alpha(D\hat{w}_q^{-1}) \cdot (D\hat{w}_q)(D_q - zA)\hat{w}_q^{-1} \\
&\quad -(D\hat{w}_q)(D_q - zA)\hat{w}_q^{-1} \cdot \hat{w}_q E_\alpha \hat{w}_q^{-1} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
R_\alpha R_\beta &= \hat{w}_q E_\alpha \hat{w}_q^{-1} \cdot \hat{w}_q E_\beta \hat{w}_q^{-1} \\
&= \hat{w}_q E_\alpha E_\beta \hat{w}_q^{-1} \\
&= \delta_{\alpha\beta} R_\beta.
\end{aligned}$$

□

Note $B_{k\alpha} = (z^k R_\alpha)_+$, $\bar{B}_{k\alpha} = (z^k R_\alpha)_-$.

Definition 1. *The q AKNS-D hierarchy is defined in the Lax pair form as follows:*

$$\begin{aligned}
L_q \hat{w}_q &= (D\hat{w}_q)(D_q - zA) \\
\partial_{k\alpha} \hat{w}_q &= -\bar{B}_{k\alpha} \hat{w}_q.
\end{aligned} \tag{34}$$

Lemma 5. *From the above definition, we have*

$$\partial_{k\alpha} L_q = [B_{k\alpha}, L_q]_q.$$

Proof. Since $L_q = (D\hat{w}_q)(D_q - zA)(\hat{w}_q^{-1})$, then

$$\begin{aligned}
\partial_{k\alpha} L_q &= (\partial_{k\alpha} D\hat{w}_q)(D_q - zA)\hat{w}_q^{-1} + (D\hat{w}_q)(D_q - zA)(\partial_{k\alpha}\hat{w}_q^{-1}) \\
&= -(D\bar{B}_{k\alpha}) \cdot (D\hat{w}_q)(D_q - zA)(\hat{w}_q^{-1}) + (D\hat{w}_q)(D_q - zA)\hat{w}_q^{-1}\bar{B}_{k\alpha} \\
&= [-\bar{B}_{k\alpha}, L_q]_q \\
&= [B_{k\alpha} - z^k R_\alpha, L_q]_q \\
&= [B_{k\alpha}, L_q]_q.
\end{aligned}$$

□

Lemma 6.

$$\partial_{k\alpha} R_\beta = [B_{k\alpha}, R_\beta].$$

Proof.

$$\begin{aligned}\partial_{k\alpha} R_\beta &= \partial_{k\alpha}(\hat{w}_q E_\beta)\hat{w}_q^{-1} \\ &= -\bar{B}_{k\alpha}\hat{w}_q E_\beta \hat{w}_q^{-1} + \hat{w}_q E_\beta \hat{w}_q^{-1}\bar{B}_{k\alpha} \\ &= -[\bar{B}_{k\alpha}, R_\beta] \\ &= [B_{k\alpha}, R_\beta].\end{aligned}$$

□

Using this relation, we can easily prove that

$$\partial_{k\alpha} B_{l\beta} - \partial_{l\beta} B_{k\alpha} = [B_{k\alpha}, B_{l\beta}]. \tag{35}$$

Lemma 7. *Every R can be fixed by its zero-order term and R_α form a basis of $\Re$.*

Proof. Using equation (30), we can find that $R^{(0)}$ must be a diagonal matrix. (Here we require that all functions we deal with are bounded and continuous everywhere as a function of x.) Because of equation (29), we can solve every $R^{(j)}$ order by order. The only freedom left is the constant diagonal part of $R^{(j)}$ which can be chosen to be zero. So the linear independent solutions are those whose zero-order term is E_α, that is why R_α form a basis of $\Re$. □

Definition 2. *The q AKNS-D hierarchy is defined as*

$$\partial_{k\alpha} L_q = [B_{k\alpha}, L_q]_q.$$

Theorem 1. *Definitions 1 and 2 are equivalent.*

Proof. Lemma 5 show that definition 1 can lead to definition 2. Now, we want to prove the converse direction. For any α, β,

$$[\partial_{k\alpha} R_\beta - [B_{k\alpha}, R_\beta], L_q]_q = [\partial_{k\alpha} R_\beta, L_q]_q - [[B_{k\alpha}, R_\beta], L_q]_q.$$

But

$$\begin{aligned}[\partial_{k\alpha} R_\beta, L_q]_q &= \partial_{k\alpha}[R_\beta, L_q]_q - [R_\beta, \partial_{k\alpha} L_q]_q \\ &= -[R_\beta, \partial_{k\alpha} L_q]_q \\ &= -[R_\beta, [B_{k\alpha}, L_q]_q]_q \\ &= [[B_{k\alpha}, R_\beta], L_q]_q\end{aligned}$$

which gives

$$[\partial_{k\alpha} R_\beta - [B_{k\alpha}, R_\beta], L_q]_q = 0$$

which means $\partial_{k\alpha} R_\beta - [B_{k\alpha}, R_\beta]$ is a resolvent. We have known that all R_α form a basis of $\Re$. Then we can express $\partial_{k\alpha} R_\beta - [B_{k\alpha}, R_\beta]$ by

$$\partial_{k\alpha} R_\beta - [B_{k\alpha}, R_\beta] = \sum_\alpha c_\alpha(z) R_\alpha.$$

Taking $\gamma \neq \beta$, we have

$$\begin{aligned}(\partial_{k\alpha} R_\beta - [B_{k\alpha}, R_\beta])R_\gamma &= -R_\beta \partial_{k\alpha} R_\gamma + R_\beta B_{k\alpha} R_\gamma \\ &= -R_\beta \partial_{k\alpha} R_\gamma + R_\beta B_{k\alpha} R_\gamma - R_\beta R_\gamma B_{k\alpha} \\ &= -R_\beta(\partial_{k\alpha} R_\gamma - [B_{k\alpha}, R_\gamma]) \\ &= -\tilde{c}_\beta R_\beta.\end{aligned}$$

On the other hand, we also know

$$(\partial_{k\alpha}R_\beta - [B_{k\alpha}, R_\beta])R_\gamma = \sum_\alpha c_\alpha R_\alpha R_\gamma = c_\gamma R_\gamma.$$

Because R_γ, R_β are linear independent, we get $\tilde{c}_\beta = c_\gamma = 0$, so $\partial_{k\alpha}R_\beta - [B_{k\alpha}, R_\beta] = c(z)R_\beta$. But

$$\begin{aligned}(\partial_{k\alpha}R_\beta - [B_{k\alpha}, R_\beta])R_\beta &= \partial_{k\alpha}R_\beta^2 - R_\beta\partial_{k\alpha}R_\beta - B_{k\alpha}R_\beta + R_\beta B_{k\alpha}R_\beta \\ &= \partial_{k\alpha}R_\beta - B_{k\alpha}R_\beta + R_\beta B_{k\alpha} - R_\beta\partial_{k\alpha}R_\beta - R_\beta^2 B_{k\alpha} + R_\beta B_{k\alpha}R_\beta \\ &= \partial_{k\alpha}R_\beta - [B_{k\alpha}, R_\beta] - R_\beta(\partial_{k\alpha}R_\beta - [B_{k\alpha}, R_\beta]) \\ &= 0.\end{aligned}$$

So we get

$$\partial_{k\alpha}R_\beta - [B_{k\alpha}, R_\beta] = 0.$$

Based on this result, it is easy to show

$$\partial_{k\alpha}B_{l\beta} - \partial_{l\beta}B_{k\alpha} = [B_{k\alpha}, B_{l\beta}]. \tag{36}$$

Following the standard method in [13], we can extend the operator $\partial_{k\alpha}$ on w_q by requiring $\partial_{k\alpha}w_q = B_{k\alpha}w_q$ and equation (36) guarantees $[\partial_{k\alpha}, \partial_{l\beta}] = 0$ holds, which means this extension is well defined, and so we prove the converse direction. □

3.3. Baker function and bilinear identity of the q AKNS-D hierarchy

Define the Baker function as

$$w_q = \hat{w}_q \cdot \exp_q(zAx)\exp\left(\sum_{k=1}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right). \tag{37}$$

Lemma 8.

$$\begin{aligned} &L_q(w_q) = 0 \\ &\partial_{k\alpha}w_q = B_{k\alpha}w_q. \end{aligned} \tag{38}$$

Proof. Using equation (5), we get

$$\begin{aligned}L_q(w_q) &= L_q\left(\hat{w}_q \cdot \exp_q(zAx)\exp\left(\sum_{k=1}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right)\right) \\ &= D\hat{w}_q(D_q - zA)\exp_q(zAx)\exp\left(\sum_{k=1}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right) \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\partial_{k\alpha}w_q &= \partial_{k\alpha}\left(\hat{w}_q \cdot \exp_q(zAx)\exp\left(\sum_{k=1}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right)\right) \\ &= -\bar{B}_{k\alpha}\hat{w}_q \cdot \exp_q(zAx)\exp\left(\sum_{k=1}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right) \\ &\quad + \hat{w}_q z^k E_\alpha \exp_q(zAx)\exp\left(\sum_{k=1}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right) \\ &= -\bar{B}_{k\alpha}\cdot w_q + z^k R_\alpha \cdot w_q \\ &= B_{k\alpha}\cdot w_q.\end{aligned}$$

□

Theorem 2 (Bilinear identity).

(i) If w_q is a solution of equation (38), it satisfies the following identity:

$$\mathrm{res}_z\Big(z^l\big(D_q^m \partial_{k\alpha}^{[\lambda]} w_q\big)\cdot w_q^{-1}\Big) = 0 \tag{39}$$

for $l = 0, 1, 2, \ldots, m = 0, 1$, $\forall[\lambda]$ (where $\partial_{k\alpha}^{[\lambda]} = \partial_{k_1\alpha_1}\partial_{k_2\alpha_2}\ldots\partial_{k_s\alpha_s}$, and $(k_1, \alpha_1), \ldots, (k_s, \alpha_s)$ is any set of indices).

(ii) If

$$w_q = \left(I + \sum_{i=1}^{\infty} w_i z^{-i}\right)\exp_q(zAx)\exp\left(\sum_{k=1}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right)$$

$$w_q^* = \left(I + \sum_{i=1}^{\infty} w_i^* z^{-i}\right)\exp_{1/q}(-zAx)\exp\left(-\sum_{k=1}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right)$$

and they satisfy the following identity:

$$\mathrm{res}_z(z^l D_q^m \partial_{k\alpha}^{[\lambda]} w_q \cdot (w_q^*)^t) = 0 \tag{40}$$

for $l = 0, 1, 2, \ldots, m = 0, 1$, $\forall[\lambda]$, then we have

(1) $(w_q^{-1})^t = w_q^*$.

(2) w_q is a solution of equation (38).

Proof.

(i) If w_q is a solution of the q AKNS hierarchy, we have

$$\partial_{k\alpha} w_q = B_{k\alpha} w_q.$$

So $\partial_{k\alpha}^{[\lambda]} w_q = f(B_{k\alpha})\cdot w_q$, where $f(B_{k\alpha})$ is a differential polynomial of $B_{k\alpha}$. It is easy to see that $(f(B_{k\alpha}))_+ = f(B_{k\alpha})$. Furthermore, from equation (38), we also know $D_q w_q = (zA - U)\cdot w_q$. Then for $\forall l \geqslant 0$ and $\forall[\lambda]$

$$\begin{aligned}
&\mathrm{res}_z(z^l D_q \partial_{k\alpha}^{[\lambda]} w_q \cdot (w_q)^{-1}) \\
&\quad = \mathrm{res}_z(z^l (D_q f(B_{k\alpha})) w_q \cdot w_q^{-1}) + \mathrm{res}_z(z^l (Df(B_{k\alpha}))\cdot (zA - U) w_q \cdot w_q^{-1}) \\
&\quad = \mathrm{res}_z(z^l (D_q f(B_{k\alpha})) + z^l (Df(B_{k\alpha}))(zA - U)) \\
&\quad = 0.
\end{aligned}$$

(ii) First, choosing $m = [\lambda] = 0$, equation (40) gives

$$\mathrm{res}_z(z^l w_q \cdot (w_q^*)^t) = 0$$

for $\forall l \geqslant 0$. This means that $w_q \cdot (w_q^*)^t$ does not contain a negative power term. From the formal expression of w_q and w_q^*, we know that it also does not contain positive power terms and the zero-order term is I, so we get $(w_q^{-1})^t = w_q^*$.

Second, from the definition of the Baker function, we obtain

$$\begin{aligned}
\partial_{k\alpha} w_q - B_{k\alpha} w_q &= (\partial_{k\alpha}\hat{w}_q)\exp_q(zAx)\exp\left(\sum_{k=1}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right) \\
&\quad + \hat{w}_q \exp_q(zAx)\exp\left(\sum_{k=1}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right) z^k E_\alpha - (z^k R_\alpha)_+ w_q \\
&= (\partial_{k\alpha}\hat{w}_q)\exp_q(zAx)\exp\left(\sum_{k=1}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right) + z^k w_q E_\alpha w_q^{-1}\cdot w_q - (z^k R_\alpha)_+ w_q \\
&= (\partial_{k\alpha}\hat{w}_q + (z^k R_\alpha)_- \hat{w}_q)\exp_q(zAx)\exp\left(\sum_{k=1}^{\infty}\sum_{\alpha=1}^{n} z^k E_\alpha t_{k\alpha}\right).
\end{aligned}$$

From equation (40), we have

$$\mathrm{res}_z(z^l(\partial_{k\alpha} - B_{k\alpha})w_q \cdot (w_q^*)^t) = 0.$$

But

$$(\partial_{k\alpha} - B_{k\alpha})w_q \cdot w_q^{-1} = (\partial_{k\alpha}\hat{w}_q + (z^k R_\alpha)_-\hat{w}_q) \cdot \hat{w}_q^{-1}.$$

So we can see it contains no positive power terms and the bilinear identity is

$$\mathrm{res}_z(z^l(\partial_{k\alpha}\hat{w}_q + (z^k R_\alpha)_-\hat{w}_q) \cdot \hat{w}_q^{-1}) = 0$$

for $\forall l \geqslant 0$, which yields

$$\partial_{k\alpha}\hat{w}_q + (z^k R_\alpha)_-\hat{w}_q = 0.$$

This equation is equivalent to equation

$$\partial_{k\alpha} w_q = B_{k\alpha} w_q.$$

Define $L_q = (Dw_q)D_q w_q^{-1}$. Simple calculation gives

$$L_q = D_q - (D_q w_q) \cdot w_q^{-1}.$$

Using the formal expression of w_q, direct calculation gives the highest-order term of $D_q w_q \cdot w_q^{-1}$ is zA and the bilinear identity (39) guarantees that $D_q w_q \cdot w_q^{-1}$ does not contain negative power terms. We note the zero-order term of $D_q w_q \cdot w_q^{-1}$ as $-U$. Then we get

$$L_q = D_q + zA - U.$$

So such w_q satisfies equation (38). □

4. The τ function of the q AKNS-D hierarchy

In [10], Iliev gives us a way to construct the τ function of the q KP hierarchy. The main idea is to 'q-shift' the time variables $t_{k\alpha}$ of the classical KP hierarchy's τ function and to prove that the Baker function constructed from this kind of τ function satisfies the bilinear identity of the q KP hierarchy. In this section, we will generalize Iliev's method to the q AKNS-D hierarchy.

Definition 3. *The q shift of $t_{k\alpha}$ is defined as*

$$t_{k\alpha} \longmapsto t_{k\alpha} + \frac{(1-q)^k}{k(1-q^k)}(a_\alpha x)^k$$

and also as $t + [Ax]_q$, we note for convenience.

Definition 4. *A matrix function $\tau(t)$ is called a τ function of the q-AKNS-D hierarchy if it satisfies*

$$\begin{aligned} \hat{w}_{\alpha\beta}(t,z) &= z^{-1}\frac{\tau_{\alpha\beta}(\ldots, t_{k\beta} - \frac{1}{k}z^{-k}, \ldots)}{\tau_{\alpha\alpha}(t)} \\ \hat{w}_{\alpha\alpha}(t,z) &= \frac{\tau_{\alpha\alpha}(\ldots, t_{k\alpha} - \frac{1}{k}z^{-k}, \ldots)}{\tau_{\alpha\alpha}(t)} \end{aligned} \tag{41}$$

where $\hat{w}(t;z)$ is a wavefunction of the q AKNS-D hierarchy.

Theorem 3 (The τ function of the q AKNS-D hierarchy). *If $\tau(t)$ is a τ function of the classical AKNS-D hierarchy, then*

$$\tau_q(t;\ x) := \tau(t + [Ax]_q)$$

is a τ function of the q AKNS-D hierarchy.

Proof. In [14], Dickey gives some τ functions of the AKNS-D hierarchy. To construct the Baker function, they have the following form:

$$\hat{w}_{W\alpha\beta}(t,z) = z^{-1}\frac{\tau_{W\alpha\beta}(\ldots, t_{k\beta} - \frac{1}{k}z^{-k}, \ldots)}{\tau_W(t)}$$
$$\hat{w}_{W\alpha\alpha}(t,z) = \frac{\tau_W(\ldots, t_{k\alpha} - \frac{1}{k}z^{-k}, \ldots)}{\tau_W(t)} \tag{42}$$

which is similar to the KP hierarchy (more details can be found in [14–16]).

Since we have defined the τ_q function as

$$\tau_q(t;\ x) := \tau(t + [Ax]_q)$$

using definition 4, equation (37) and lemma 2, it is easy to see that the function w_q is just the q shift of the classical Baker function, i.e.

$$w_q(t;\ x) = w(t + [Ax]_q).$$

The classical Baker function w satisfies the classical bilinear identity

$$\mathrm{res}_z(z^l(\partial_{k\alpha}^{[\lambda]} w)\cdot w^{-1}) = 0 \qquad \text{for} \quad \forall\, l \geqslant 0, \quad \forall\, [\lambda]. \tag{43}$$

We want to show that the function w_q satisfies the q bilinear identity

$$\mathrm{res}_z(z^l(D_q\partial_{k\alpha}^{[\lambda]} w_q)\cdot w_q^{-1}) = 0.$$

Submitting the expression for w_q into the q bilinear identity, we get

$$\begin{aligned}
&\mathrm{res}_z(z^l(D_q\partial_{k\alpha}^{[\lambda]} w(t+[Ax]_q))\cdot w^{-1}(t+[Ax]_q)\\
&\quad = [\mathrm{res}_z(z^l(\partial_{k\alpha}^{[\lambda]} w(t+[Aqx]_q))\cdot w^{-1}(t+[Ax]_q)\\
&\quad\quad - \mathrm{res}_z(z^l(\partial_{k\alpha}^{[\lambda]} w(t+[Ax]_q))\cdot w^{-1}(t+[Ax]_q)]\cdot\frac{1}{x(q-1)}.
\end{aligned} \tag{44}$$

Taking $t' = t + [Ax]_q$, the classical bilinear identity equation (43) shows that the second term on the right-hand side of the above equation is zero.

The first term of equation (44) is

$$\frac{1}{x(q-1)}[\mathrm{res}_z(z^l(\partial_{k\alpha}^{[\lambda]} w(t+[Aqx]_q))\cdot w^{-1}(t+[Ax]_q)].$$

For convenience, we denote $\frac{(1-q)^k}{k(1-q^k)}(a_\alpha qx)^k$ as $x_{k\alpha}^q$ and $\frac{(1-q)^k}{k(1-q^k)}(a_\alpha x)^k$ as $x_{k\alpha}$. Using the Taylor expansion of $w(t + [Ax]_q)$ at $t + [Ax]_q$, we get

$$\begin{aligned}
&\frac{1}{x(q-1)}[\mathrm{res}_z(z^l(\partial_{k\alpha}^{[\lambda]} w(t+[Aqx]_q))\cdot w^{-1}(t+[Ax]_q))]\\
&\quad = \frac{1}{x(q-1)}\Bigg[\mathrm{res}_z(z^l(\partial_{k\alpha}^{[\lambda]} w(t+[Ax]_q))\cdot w^{-1}(t+[Ax]_q))\\
&\quad\quad + \sum_{l,\beta,[\eta]}\mathrm{res}_z(z^l(\partial_{k\alpha}^{[\lambda]}\partial_{l\beta}^{[\eta]} w(t+[Ax]_q))\cdot w^{-1}(t+[Ax]_q))\cdot(x_{l\beta}^q - x_{l\beta})^{[\eta]}\Bigg].
\end{aligned} \tag{45}$$

Every term in the above equation has the following form:

$$\frac{(x_{l\beta}^q - x_{l\beta})^{[\eta]}}{x(q-1)}\mathrm{res}_z(z^l(\partial_{k\alpha}^{[\lambda]}\partial_{l\beta}^{[\eta]} w(t+[Ax]_q))\cdot w^{-1}(t+[Ax]_q))$$

where the $\partial_{l\beta}^{[\eta]}$ and the $(x_{l\beta}^q - x_{l\beta})^{[\eta]}$ come from the Taylor expansion. The classical bilinear identity of the AKNS hierarchy makes sure that this kind of term is zero, for any $[\lambda]$, $[\eta]$ and

$l \geqslant 0$, so we find that the first term of equation (44) is also zero. That means the Baker function w_q satisfies the q bilinear identity. □

Furthermore, comparing both sides of the Taylor expansion of equation (44), we can see that the D_q and the $\partial_{k\alpha}$ are not independent, and we have the following relation:

$$D_q = \frac{1}{x(q-1)} \sum_{l,\beta,\eta} c([\eta])(x_{l\beta}^q - x_{l\beta})^{[\eta]} \partial_{l\beta}^{[\eta]}$$
$$= \sum_{\beta} a_\beta \partial_{1\beta} + \mathrm{O}(q-1) \tag{46}$$

where $c([\eta])$ is the constant coming from the Taylor expansion. When $q \to 1$, we can get the relation

$$\partial = \sum_{\beta} a_\beta \partial_{1\beta}$$

which holds for the classical AKNS hierarchy.

In this paper, we have presented some results which may be regarded as a preliminary step to gaining a better understanding of the q deformation of a classical integrable system. Many questions remain to be addressed. Perhaps the most straightforward one is whether it is possible to generalize the present work to the mcKP hierarchy, because Dickey has pointed out [15] that the AKNS-D hierarchy is a special case of the mcKP hierarchy. Furthermore, the Grassmannian of the q deformation should also be considered. We will consider those problems in future papers.

Acknowledgment

The present work is supported by the Scientific Foundation of Chinese Academy of Science, the NSF of China and '973' Project of China.

References

[1] Bobenko A I and Seiler R (ed) 1999 *Discrete Integrable Geometry and Physics* (Oxford: Oxford University Press)
[2] Zhang D H 1993 *J. Phys. A: Math. Gen.* **26** 2389
[3] Wu Z Y, Zhang D H and Zheng Q R 1994 *J. Phys. A: Math. Gen.* **27** 5307
[4] Mac J and Seco M 1996 *J. Math. Phys.* **37** 6510
[5] Frenkel E 1996 *Int. Math. Res. Notice* **2** 55
[6] Frenkel E and Reshetkhin N 1996 *Commun. Math. Phys.* **178** 237
[7] Khesin B, Lyubashenko V and Roger C 1997 *J. Funct. Anal.* **143** 55
[8] Adler M, Horozov E and van Moerbeke P 1998 *Phys. Lett.* A **242** 139
[9] Haine L and Iliev P 1997 *J. Phys. A: Math. Gen.* **30** 7217
[10] Iliev P 1998 *Lett. Math. Phys.* **44** 187
[11] Tu M H, Shaw J C and Lee C R 1999 *Lett. Math. Phys.* **49** 33
[12] Tu M H 1999 *Lett. Math. Phys.* **49** 95
[13] Dickey L A 1991 *Soliton Equations and Hamiltonian System (Advanced Series in Mathematical Physics vol 12)* (Singapore: World Scientific)
[14] Dickey L A 1991 *J. Math. Phys.* **32** 2996
[15] Dickey L A 1993 On Segal-Wilson's definition of the τ-function and hierarchy AKNS-D and mcKP *Integrable Systems: The Verdier Memerial Conf.* ed O Babelon, P Cartier and Y Kosmann-Schwarzbach (Cambridge, MA: Birkhauser Boston)
[16] Segal G and Wilson G 1985 *Publ. Math. IHES* **63** 1
[17] Gasper G and Rahman M 1990 Basic hypergeometric series *Encyclopedia Math. Appl.* vol 35 (Cambridge: Cambridge University Press) chapter 1